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Vertex operators and soliton solutions of affine Toda model with $U(2)$ symmetry

I Cabrera-Carnero, J F Gomes, G M Sotkov and A H Zimerman

Instituto de Física Teórica, IFT/UNESP, Rua Pamplona 145, 01405-900, São Paulo, SP, Brazil

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Abstract

The symmetry structure of the non-Abelian affine Toda model based on the coset $SL(3)/SL(2) \otimes U(1)$ is studied. It is shown that the model possess non-Abelian Noether symmetry closing into a q-deformed $SL(2) \otimes U(1)$ algebra. Specific two-vertex soliton solutions are constructed.

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1. Introduction

The family of 2D relativistic integrable models (IM) known as affine Toda field theories (ATFTs) have been intensively studied due to their relation to certain deformations of 2d conformal field theories (CFT) [1–4] as well as due to their rich soliton spectrum [5–7]. Certain integrable perturbations of $SU(N)$-WZW models and their gauged versions related to ATFTs have also important applications in the condensed mATFTer problems, for example in the description of Heisenberg antiferromagnetic spin chains and ladders [8].

According to their symmetries we distinguish two classes of ATFTs: *Abelian* (A) and *non-Abelian* (NA) ones. The main feature of the non-Abelian ATFTs is that they manifest local or global Noether symmetries (say, $U(1)^{\otimes l}$, $SL(2) \otimes U(1)$, etc), while the Abelian ones do not possess any other symmetries except the discrete Z_n in the case of imaginary coupling. As a consequence the NA-ATFTs admit topological solitons carrying certain Noether charges as well. Examples of electrically charged ($U(1)$ or $U(1) \otimes U(1)$) topological solitons have been constructed in [9–12]. As is well known [13], such finite energy classical solutions play a crucial role in the semiclassical quantization as well as for establishing their strong-coupling particle spectra [14]. The exact quantum S-matrices of certain T-self-dual NA Toda models [15] with $U(1)$ symmetry have been derived in [16].

Although the general theory for constructing non-Abelian affine Toda theories is well developed in terms of graded affine algebras (see, for instance, [17, 18] and references therein), its application to specific NA-ATFT possessing non-Abelian symmetries gives rise to certain unexpected and interesting structures. One of them is related to the existence of pairs of T-dual NA-ATFT models described in [19]. In this paper we will discuss a q-deformed algebraic structure that appears in the simplest integrable model of this type, based on the coset $SL(3)/SL(2) \otimes U(1)$, constructed in section 2 (see also section 6 of [11]). In section 3 we show that it is invariant under specific non-local and non-Abelian transformations and the corresponding Noether charges close a q-deformed $SL(2, R) \otimes U(1)$ Poisson bracket algebra. As a consequence its soliton solutions carry both $U(1)$ and isospin charges. The goal of this paper is to explicitly construct these 1-soliton solutions and to investigate their internal non-Abelian symmetries. The two-vertex soliton solutions obtained in sections 4 and 5 by the dressing method represent a specific subclass of 1-solitons whose spectrum depends on one real parameter only. Their masses and charges are derived in section 5.

2. Affine Toda models with non-Abelian symmetries

2.1. The model in the group G_0

As is well known [18], the generic NA Toda models are classified according to a $\mathcal{G}_0 \subset \mathcal{G}$ embedding induced by the grading operator Q decomposing a finite- or infinite-dimensional Lie algebra $\mathcal{G} = \oplus_i \mathcal{G}_i$ where $[Q, \mathcal{G}_i] = \mathrm{i}\mathcal{G}_i$ and $[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}$. A group element g can then be written in terms of the Gauss decomposition as

$$g = NBM \tag{2.1}$$

where $N = \exp \mathcal{G}_<$, $B = \exp \mathcal{G}_0$ and $M = \exp \mathcal{G}_>$. The physical fields lie in the zero grade subgroup B and the models we seek correspond to the coset $H_-\backslash G/H_+$, for $H_\pm$ generated by positive/negative grade operators.

For consistency with the Hamiltonian reduction formalizm, the phase space of the G-invariant WZW model is reduced by specifying the constant generators $\epsilon_\pm$ of grade ± 1. In order to derive an action for B, invariant under

$$g \longrightarrow g' = \alpha_- g \alpha_+ \tag{2.2}$$

where $\alpha_\pm(z, \bar{z})$ lie in the positive/negative grade subgroup we have to introduce a set of auxiliary gauge fields $A \in \mathcal{G}_<$ and $\bar{A} \in \mathcal{G}_>$ transforming as

$$A \longrightarrow A' = \alpha_- A \alpha_-^{-1} + \alpha_- \partial \alpha_-^{-1} \qquad \bar{A} \longrightarrow \bar{A}' = \alpha_+^{-1} \bar{A} \alpha_+ + \bar{\partial} \alpha_+^{-1} \alpha_+ \tag{2.3}$$

where $\partial = \partial_t + \partial_x$, $\bar{\partial} = \partial_t - \partial_x$. The resulting action is the $G/H (= H_-\backslash G/H_+)$ gauged WZW

$$S_{G/H}(g, A, \bar{A}) = S_{\mathrm{WZW}}(g) - \frac{k}{4\pi} \int \mathrm{d}^2x \, \mathrm{Tr}(A(\bar{\partial} g g^{-1} - \epsilon_+) + \bar{A}(g^{-1}\partial g - \epsilon_-) + A g \bar{A} g^{-1}).$$

Since the action $S_{G/H}$ is H-invariant, we may choose $\alpha_- = N^{-1}$ and $\alpha_+ = M^{-1}$. From the orthogonality of the graded subpaces, i.e. $\mathrm{Tr}(\mathcal{G}_i \mathcal{G}_j) = 0$, $i + j \neq 0$, we find

$$\begin{aligned} S_{G/H}(g, A, \bar{A}) &= S_{G/H}(B, A', \bar{A}') \\ &= S_{\mathrm{WZW}}(B) - \frac{k}{4\pi} \int \mathrm{d}^2x \, \mathrm{Tr}[-A'\epsilon_+ - \bar{A}'\epsilon_- + A' B \bar{A}' B^{-1}] \end{aligned} \tag{2.4}$$

where we have introduced the WZW model action

$$S_{\mathrm{WZW}} = -\frac{k}{4\pi} \int \mathrm{d}^2x \, \mathrm{Tr}(g^{-1}\partial g g^{-1} \bar{\partial} g) + \frac{k}{24\pi} \int_D \epsilon^{ijk} \, \mathrm{Tr}(g^{-1}\partial_i g g^{-1} \partial_j g g^{-1} \partial_k g) \, \mathrm{d}^3x. \tag{2.5}$$

Performing the integration over the auxiliary fields A and $\bar{A}$, we find the effective action

$$S = S_{\mathrm{WZW}}(B) - \frac{k}{2\pi} \int \mathrm{Tr}(\epsilon_+ B \epsilon_- B^{-1}) \, \mathrm{d}^2x \tag{2.6}$$

which describes integrable perturbations of the $\mathcal{G}_0$-WZNW model. These perturbations are classified in terms of the possible constant grade ±1 operators $\epsilon_\pm$. The equations of motion associated with action (2.6) are

$$\bar\partial(B^{-1}\partial B) + [\epsilon_-, B^{-1}\epsilon_+ B] = 0 \qquad \partial(\bar\partial B B^{-1}) - [\epsilon_+, B\epsilon_- B^{-1}] = 0. \quad (2.7)$$

For the $\mathcal{G} = \hat{SL}(3)$ case with homogeneous gradation, $Q = d$ and $\epsilon_\pm = \mu\lambda_2 \cdot H^{(\pm1)}$ and $B = nam$, where

$$n = \mathrm{e}^{\tilde\chi_1 E_{-\alpha_1}}\, \mathrm{e}^{\tilde\chi_2 E_{-\alpha_2}}\, \mathrm{e}^{\tilde\chi_3 E_{-\alpha_1-\alpha_2}} \qquad a = \mathrm{e}^{R_1\lambda_1\cdot H + R_2\lambda_2\cdot H} \qquad m = \mathrm{e}^{\tilde\psi_1 E_{\alpha_1}}\, \mathrm{e}^{\tilde\psi_2 E_{\alpha_2}}\, \mathrm{e}^{\tilde\psi_3 E_{\alpha_1+\alpha_2}} \quad (2.8)$$

we find the explicit form for the action

$$S_{\mathrm{eff}} = -\frac{k}{8\pi}\int \mathrm{d}z\,\mathrm{d}\bar z\left(\frac{1}{3}(2\partial R_1\bar\partial R_1 - \partial R_1\bar\partial R_2 - \partial R_2\bar\partial R_1 + 2\partial R_2\bar\partial R_2) + 2\partial\tilde\chi_1\bar\partial\tilde\psi_1\,\mathrm{e}^{R_1}\right.$$
$$\left. + 2\partial\tilde\chi_2\bar\partial\tilde\psi_2\,\mathrm{e}^{R_2} + 2(\partial\tilde\chi_3 - \tilde\chi_2\partial\tilde\chi_1)(\bar\partial\tilde\psi_3 - \tilde\psi_2\bar\partial\tilde\psi_1)\,\mathrm{e}^{R_1+R_2} - V\right). \quad (2.9)$$

where $V = \mu^2\big(\lambda_2^2 + \tilde\psi_2\tilde\chi_2\,\mathrm{e}^{R_2} + \tilde\psi_3\tilde\chi_3\,\mathrm{e}^{R_1+R_2}\big)$.

2.2. Reduction to the coset G_0/G_0^0

We now introduce the subalgebra $\mathcal{G}_0^0$ such that $\big[\mathcal{G}_0^0, \epsilon_\pm\big] = 0$ as an additional ingredient which characterizes the symmetry of action (2.6) under chiral transformations

$$B' = \bar\Omega(\bar z)B\Omega(z) \qquad \bar\Omega, \Omega \in G_0^0. \quad (2.10)$$

As a consequence of symmetry under (2.10), the following chiral conservation laws are derived from the equations of motion (2.7)

$$\bar\partial\,\mathrm{Tr}(XB^{-1}\partial B) = \partial\,\mathrm{Tr}(X\bar\partial B B^{-1}) = 0 \qquad X \in \mathcal{G}_0^0. \quad (2.11)$$

In order to reduce the model to the coset G_0/G_0^0, we impose the subsidiary constraints

$$J_X = \mathrm{Tr}(XB^{-1}\partial B) = 0 \qquad \bar J_X = \mathrm{Tr}(X\bar\partial B B^{-1}) = 0 \qquad X \in \mathcal{G}_0^0 \quad (2.12)$$

which can be incorporated in the action by introducing the auxiliary gauge fields $A^{(0)}, \bar A^{(0)} \in \mathcal{G}_0^0$. For the models where $\mathcal{G}_0^0 = U(1)$, [9, 10] or $\mathcal{G}_0^0 = U(1)\otimes U(1)$, [11], the action was constructed imposing symmetry under axial transformations

$$B'' = \alpha_0(\bar z, z)B\alpha_0(\bar z, z) \qquad \alpha_0 \in G_0^0$$

and

$$A''^{(0)} = A^{(0)} - \alpha_0^{-1}\partial\alpha_0 \qquad \bar A''^{(0)} = \bar A^{(0)} - \alpha_0^{-1}\bar\partial\alpha_0. \quad (2.13)$$

For a general non-Abelian $\mathcal{G}_0^0$ we can define a second grading structure $Q' = \lambda_1 \cdot H$ which decomposes $\mathcal{G}_0^0$ into positive, zero and negative subspaces, i.e., $\mathcal{G}_0^0 = \mathcal{G}_0^{0,<} \oplus \mathcal{G}_0^{0,0} \oplus \mathcal{G}_0^{0,>}$. Following the same principle as in [9–11] we seek an action invariant under

$$B'' = \gamma_0(\bar z, z)\gamma_-(\bar z, z)B\gamma_+(\bar z, z)\gamma_0(\bar z, z) \qquad \gamma_0 \in G_0^{0,0} \qquad \gamma_- \in G_0^{0,<} \qquad \gamma_+ \in G_0^{0,>}$$

and choose $\gamma_0(\bar z, z), \gamma_-(\bar z, z), \gamma_+(\bar z, z) \in G_0^0$ such that $B'' = \gamma_0\gamma_- B\gamma_+\gamma_0 = g_0^f \in G_0/G_0^0$. Note that B is decomposed into the Gauss form according to the second grading structure Q'. Denote $\Gamma_- = \gamma_0\gamma_-$ and $\Gamma_+ = \gamma_+\gamma_0$. Then the action

$$S(B, A^{(0)}, \bar A^{(0)}) = S\big(g_0^f, A'^{(0)}, \bar A'^{(0)}\big) = S_{\mathrm{WZW}}(B) - \frac{k}{2\pi}\int \mathrm{Tr}(\epsilon_+ B\epsilon_- B^{-1})\,\mathrm{d}^2x$$
$$- \frac{k}{2\pi}\int \mathrm{Tr}\big(A^{(0)}\bar\partial B B^{-1} + \bar A^{(0)} B^{-1}\partial B + A^{(0)} B\bar A^{(0)} B^{-1} + A_0^{(0)}\bar A_0^{(0)}\big)\,\mathrm{d}^2x \quad (2.14)$$

is invariant under the transformations $B' = \Gamma_- B \Gamma_+$,

$$\begin{aligned} A_0'^0 &= A_0^{(0)} - \gamma_0^{-1}\partial\gamma_0 & \bar{A}_0'^0 &= \bar{A}_0^{(0)} - \gamma_0^{-1}\bar{\partial}\gamma_0 \\ A'^{(0)} &= \Gamma_- A_{(0)}\Gamma_-^{-1} - \partial\Gamma_-\Gamma_-^{-1} & \bar{A}'^{(0)} &= \Gamma_+^{-1}\bar{A}_{(0)}\Gamma_+ - \Gamma_+^{-1}\bar{\partial}\Gamma_+ \end{aligned} \tag{2.15}$$

where $A^{(0)} = A_0^{(0)} + A_-^{(0)}$ and $\bar{A}^{(0)} = \bar{A}_0^{(0)} + \bar{A}_+^{(0)}$ and $A_0^{(0)}, \bar{A}_0^{(0)} \in \mathcal{G}_0^{0,0}$, $A_-^{(0)} \in \mathcal{G}_0^{0,<}$, $\bar{A}_+^{(0)} \in \mathcal{G}_0^{0,>}$.

Let us apply the above gauge fixing procedure for the simplest case of $\mathcal{G}_0 = SL(3, R)$, $Q = d$, $\epsilon_\pm = \mu\lambda_2 \cdot H^{(\pm 1)}$ and $\mathcal{G}_0^0 = SL(2, R) \otimes U(1)$, i.e., for IM defined on the coset $\Gamma_- \backslash SL(3)/\Gamma_+$ where $\Gamma_\pm = \exp(\tilde{\chi}_\pm E_{\pm\alpha_1}) \exp\left(\frac{1}{2}\lambda_i \cdot H R_i\right)$. Hence the auxiliary fields $A_0^{(0)}, \bar{A}_0^{(0)}, A_-^{(0)}$ and $\bar{A}_+^{(0)}$ can be parametrized as follows:

$$\begin{aligned} A_0^{(0)} &= a_1\lambda_1 \cdot H + a_2(\lambda_2 - \lambda_1)\cdot H & \bar{A}_0^{(0)} &= \bar{a}_1\lambda_1 \cdot H + \bar{a}_2(\lambda_2 - \lambda_1)\cdot H \\ A_-^{(0)} &= a_{21}E_{-\alpha_1}^{(0)} & \bar{A}_+^{(0)} &= \bar{a}_{12}E_{\alpha_1}^{(0)} \end{aligned} \tag{2.16}$$

where $a_i(z, \bar{z}), \bar{a}_i(z, \bar{z}), a_{21}(z, \bar{z})$ and $\bar{a}_{12}(z, \bar{z})$ are arbitrary functions and

$$g_0^f = \mathrm{e}^{\chi_1 E_{-\alpha_1} + \chi_2 E_{-\alpha_1-\alpha_2}}\,\mathrm{e}^{\psi_1 E_{\alpha_1} + \psi_2 E_{\alpha_1+\alpha_2}}. \tag{2.17}$$

The relation between the fields $\tilde{\psi}_i, \tilde{\chi}_i, R_i$ parametrizing the group element (2.8) and the physical fields of the gauged model $\psi_1, \chi_1, \psi_2, \chi_2$ parametrizing g_0^f is given by

$$B = \mathrm{e}^{\frac{1}{2}R_1\lambda_1\cdot H + \frac{1}{2}R_2\lambda_2\cdot H}\,\mathrm{e}^{\chi_3 E_{-\alpha_1}}\left(g_0^f\right)\mathrm{e}^{\psi_3 E_{\alpha_1}}\,\mathrm{e}^{\frac{1}{2}R_1\lambda_1\cdot H + \frac{1}{2}R_2\lambda_2\cdot H} \tag{2.18}$$

or in components,

$$\begin{aligned} \tilde{\chi}_1 &= \chi_3\,\mathrm{e}^{-\frac{1}{2}R_1} & \tilde{\psi}_1 &= \psi_3\,\mathrm{e}^{-\frac{1}{2}R_1} \\ \tilde{\chi}_2 &= \chi_2\,\mathrm{e}^{-\frac{1}{2}R_2} & \tilde{\psi}_2 &= \psi_2\,\mathrm{e}^{-\frac{1}{2}R_2} \\ \tilde{\chi}_3 &= \chi_1\,\mathrm{e}^{-\frac{1}{2}(R_1+R_2)} & \tilde{\psi}_3 &= \psi_1\,\mathrm{e}^{-\frac{1}{2}(R_1+R_2)}. \end{aligned} \tag{2.19}$$

In order to calculate the path integral over the auxiliary gauge fields (2.16) we first simplify the last term in equation (2.14)

$$\begin{aligned} &\mathrm{Tr}\left(A_0^{(0)}\bar{A}_0^{(0)} + A^{(0)}g_0^f\bar{A}^{(0)}g_0^{f-1} + A^{(0)}\bar{\partial}g_0^f g_0^{f-1} + \bar{A}^{(0)}g_0^{f-1}\partial g_0^f\right) \\ &\qquad = \bar{a}_i M_{ij} a_j + \bar{a}_i N_i + \bar{N}_i a_i + \bar{a}_{12}a_{21}(1 + \psi_2\chi_2) - \bar{a}_{12}\psi_2\partial\chi_1 - a_{21}\chi_2\bar{\partial}\psi_1 \end{aligned} \tag{2.20}$$

where we have introduced the matrix M,

$$M = \begin{pmatrix} \frac{4}{3} + \psi_1\chi_1 & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} + \psi_2\chi_2 \end{pmatrix} \tag{2.21}$$

and the vectors N and $\bar{N}$,

$$\bar{N} = (-(\bar{\partial}\psi_1 - \bar{a}_{12}\psi_2)\chi_1, -(\chi_2\bar{\partial}\psi_2)) \qquad N = \begin{pmatrix} -(\partial\chi_1 - a_{21}\chi_2)\psi_1 \\ -(\psi_2\partial\chi_2) \end{pmatrix}. \tag{2.22}$$

We first evaluate the integral over a_i and $\bar{a}_i$ in the partition function

$$Z = \int \mathcal{D}B\mathcal{D}A_0^{(0)}\mathcal{D}\bar{A}_0^{(0)}\mathcal{D}A_-^{(0)}\mathcal{D}\bar{A}_+^{(0)}\,\mathrm{e}^{-S(B,A^{(0)},\bar{A}^{(0)})} \tag{2.23}$$

i.e., the Gaussian integral

$$\int D\bar{a}Da\,\mathrm{e}^{\int(\bar{a}Ma + \bar{a}N + \bar{N}a)} = \mathrm{const}\,\mathrm{e}^{-(\bar{N}M^{-1}N)}.$$

Taking into account the explicit form of $\bar{N}M^{-1}N$,i.e.,

$$
\begin{aligned}
NM^{-1}N = {} & \frac{4\Delta}{3D}a_{21}\bar{a}_{12} + 2\frac{\chi_2 a_{21}}{3D}(\chi_2\psi_1\bar{\partial}\psi_2 - 2(1+\psi_2\chi_2)\bar{\partial}\psi_1) \\
& + 2\frac{\psi_2\bar{a}_{12}}{3D}(\psi_2\chi_1\partial\chi_2 - 2(1+\psi_2\chi_2)\partial\chi_1) \\
& + \frac{1}{3D}(-(4+3\psi_2\chi_2)\psi_1\chi_1\bar{\partial}\psi_1\partial\chi_1 - 2\chi_1\psi_2\partial\chi_2\bar{\partial}\psi_3 \\
& - 2\chi_2\psi_1\bar{\partial}\psi_2\partial\chi_1 - (4+3\psi_1\chi_1)\psi_2\chi_2\bar{\partial}\psi_2\partial\chi_2)
\end{aligned}
$$

$$
\Delta = (1+\psi_2\chi_2)^2 + \psi_1\chi_1\left(1+\frac{3}{4}\psi_2\chi_2\right)
$$

$$
D = \mathrm{Det}\, M = \frac{4}{3}\left(1+\psi_1\chi_1+\psi_2\chi_2+\frac{3}{4}\psi_1\chi_1\psi_2\chi_2\right) \tag{2.24}
$$

and equation (2.20), we integrate over a_{21} and $\bar{a}_{12}$. As a result we derive the effective action for the gauge fixed model[1]

$$
\begin{aligned}
S_{\mathrm{eff}} = -\frac{k}{2\pi}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\bigg(&\frac{1}{\Delta}(\bar{\partial}\psi_2\partial\chi_2(1+\psi_1\chi_1+\psi_2\chi_2) + \bar{\partial}\psi_1\partial\chi_1(1+\psi_2\chi_2) \\
& - \frac{1}{2}(\psi_2\chi_1\bar{\partial}\psi_1\partial\chi_2 + \chi_2\psi_1\bar{\partial}\psi_2\partial\chi_1)) - V\bigg)
\end{aligned} \tag{2.25}
$$

where $V = \mu^2\left(\frac{2}{3}+\psi_1\chi_1+\psi_2\chi_2\right)$. It appears to be the simplest generalization of the complex sine-Gordon model [23] and belongs to the same hierarchy of the Fordy–Kulish (multicomponent) nonlinear Schrödinger model [20, 22]. One can also derive it as a further Hamiltonian reduction of the $A_2^{(1)}$-homogeneous sine-Gordon model [21].

It is worthwhile mentioning that the classical integrability of the gauged fixed model (2.25) is a consequence of the integrability of the corresponding ungauged model (2.9). The zero curvature (Lax) representation of the IM (2.9)(or equivalently (2.7)) has the well-known form

$$
\partial\bar{\mathcal{A}} - \bar{\partial}\mathcal{A} - [\mathcal{A},\bar{\mathcal{A}}] = 0 \qquad \mathcal{A},\bar{\mathcal{A}} \in \oplus_{i=0,\pm1}\mathcal{G}_i \tag{2.26}
$$

with

$$
\mathcal{A} = -B\epsilon_- B^{-1} \qquad \bar{\mathcal{A}} = \epsilon_+ + \bar{\partial}BB^{-1}. \tag{2.27}
$$

We next impose the constraints (2.12) on the group element B (2.8), i.e. substituting the non-physical fields R_1, R_2, $\tilde{\psi}_1$ and $\tilde{\chi}_1$ by their nonlocal expressions obtained as a solution of the constraints (2.12) (see equations (3.34) and (3.35) for their explicit form). This gives the Lax connection $\mathcal{A}$, $\bar{\mathcal{A}}$ for the gauged model (2.25). It can easily be verified that substituting (2.27) into (2.26) and taking into account (2.12) (or (3.34) and (3.35)), one reproduces the equations of motion derived from the action (2.25). Then the existence of an infinite set (of commuting) conserved charges P_m, $m = 0, 1, \ldots$ is a simple consequence of equation (2.26), namely,

$$
P_m(t) = \mathrm{Tr}(T(t))^m \qquad \partial_t P_m = 0 \qquad T(t) = \lim_{L\to\infty}\mathcal{P}\exp\int_{-L}^{L}\mathcal{A}_x(t,x)\,\mathrm{d}x.
$$

Hence, the above described procedure for the derivation of the NA affine Toda model (2.25) as gauged G/H two loop WZW models leads to an integrable model by construction.

[1] Since in this paper we are interested in the solution and symmetries of the classical gauge fixed model we are consistently neglecting all the quantum contributions to S_{eff} coming from the determinant factors, ghost field action, counter terms, etc.

3. Symmetries

3.1. Chiral symmetries in the group G_0

Let us now discuss the symmetry structure of the ungauged IM on the $SL(3)$ group given by equation (2.9). It is generated by the chiral transformation (2.10), i.e., $B' = \bar{\Omega}(\bar{z})B\Omega(z)$, $\bar{\Omega}, \Omega \in G_0^0 = SL(2) \otimes U(1)$ generated by $\{\lambda_1 \cdot H, \lambda_2 \cdot H, E_{\pm\alpha_1}\}$. We make use of the defining representation of $SL(3)$ in terms of 3×3 matrices to parametrize the zero grade group element B (2.8) in terms of the Gauss decomposition, i.e.,

$$
\begin{aligned}
&B_{1,1} = \mathrm{e}^{\frac{2}{3}R_1+\frac{1}{3}R_2} \qquad B_{1,2} = B_{1,1}\tilde{\psi}_1 \qquad B_{1,3} = B_{1,1}\tilde{\psi}_3 \qquad B_{2,3} = B_{1,1}(\mathrm{e}^{-R_1}\tilde{\psi}_2 + \tilde{\chi}_1\tilde{\psi}_3)\\
&B_{2,1} = B_{1,1}\tilde{\chi}_1 \qquad B_{2,2} = B_{1,1}(\mathrm{e}^{-R_1} + \tilde{\psi}_1\tilde{\chi}_1) \qquad B_{3,1} = B_{1,1}\tilde{\chi}_3\\
&B_{3,2} = B_{1,1}(\mathrm{e}^{-R_1}\tilde{\chi}_2 + \tilde{\psi}_1\tilde{\chi}_3) \qquad B_{3,3} = B_{1,1}(\tilde{\psi}_1\tilde{\chi}_1 + \mathrm{e}^{-R_1}\tilde{\psi}_2\tilde{\chi}_2 + \mathrm{e}^{-R_1-R_2}).
\end{aligned}
\tag{3.28}
$$

We also find for the chiral symmetry transformations

$$
\begin{aligned}
&\Omega_{1,1} = \mathrm{e}^{\frac{2}{3}\epsilon_1+\frac{1}{3}\epsilon_2} \qquad \Omega_{1,2} = \Omega_{1,1}\epsilon_+ \qquad \Omega_{2,1} = \Omega_{1,1}\epsilon_-\\
&\Omega_{2,2} = \Omega_{1,1}(\mathrm{e}^{-\epsilon_1} + \epsilon_-\epsilon_+) \qquad \Omega_{3,3} = \Omega_{1,1}\,\mathrm{e}^{-\epsilon_1-\epsilon_2}\\
&\Omega_{1,3} = \Omega_{2,3} = \Omega_{3,2} = \Omega_{3,1} = 0
\end{aligned}
\tag{3.29}
$$

and $\bar{\Omega} = \Omega(\epsilon \to \bar{\epsilon})$ where $\epsilon = \epsilon(z)$ and $\bar{\epsilon} = \bar{\epsilon}(\bar{z})$. The infinitesimal chiral transformations for (2.10) yield the following field transformations:

$$
\begin{aligned}
&\delta R_1 = \epsilon_1 + \bar{\epsilon}_1 + 2\epsilon_-\tilde{\psi}_1 + 2\bar{\epsilon}_+\tilde{\chi}_1\\
&\delta R_2 = \epsilon_2 + \bar{\epsilon}_2 - \epsilon_-\tilde{\psi}_1 - \bar{\epsilon}_+\tilde{\chi}_1\\
&\delta\tilde{\psi}_1 = \epsilon_+ - \epsilon_1\tilde{\psi}_1 + \bar{\epsilon}_+\,\mathrm{e}^{-R_1} - \epsilon_-\tilde{\psi}_1^2\\
&\delta\tilde{\chi}_1 = \bar{\epsilon}_- - \bar{\epsilon}_1\tilde{\chi}_1 + \epsilon_-\,\mathrm{e}^{-R_1} - \bar{\epsilon}_+\tilde{\chi}_1^2\\
&\delta\tilde{\psi}_2 = \epsilon_-(\tilde{\psi}_1\tilde{\psi}_2 - \tilde{\psi}_3) - \epsilon_2\tilde{\psi}_2\\
&\delta\tilde{\chi}_2 = \bar{\epsilon}_+(\tilde{\chi}_1\tilde{\chi}_2 - \tilde{\chi}_3) - \bar{\epsilon}_2\tilde{\chi}_2\\
&\delta\tilde{\psi}_3 = -(\epsilon_1 + \epsilon_2)\tilde{\psi}_3 + \bar{\epsilon}_+\tilde{\psi}_2\,\mathrm{e}^{-R_1} - \epsilon_-\tilde{\psi}_1\tilde{\psi}_3\\
&\delta\tilde{\chi}_3 = -(\bar{\epsilon}_1 + \bar{\epsilon}_2)\tilde{\chi}_3 + \epsilon_-\tilde{\chi}_2\,\mathrm{e}^{-R_1} - \bar{\epsilon}_+\tilde{\chi}_1\tilde{\chi}_3.
\end{aligned}
\tag{3.30}
$$

As a consequence of the invariance of the action (2.9) under the chiral transformations (3.30) we find Noether currents to correspond to the chiral currents (2.11) associated with the $\mathcal{G}_0^0$ subalgebra. For the explicit example of the $\mathcal{G}_0 = SL(3)$ we have

$$
\begin{aligned}
J_{-\alpha_1} &= \partial\tilde{\psi}_1 - \tilde{\psi}_1^2\partial\tilde{\chi}_1\,\mathrm{e}^{R_1} + \partial\tilde{\chi}_2(\tilde{\psi}_1\tilde{\psi}_2 - \tilde{\psi}_3)\,\mathrm{e}^{R_2}\\
&\quad + (\partial\tilde{\chi}_3 - \tilde{\chi}_2\partial\tilde{\chi}_1)(\tilde{\psi}_1\tilde{\psi}_2 - \tilde{\psi}_3)\tilde{\psi}_1\,\mathrm{e}^{R_1+R_2} + \tilde{\psi}_1\partial R_1\\
J_{\alpha_1} &= \partial\tilde{\chi}_1\,\mathrm{e}^{R_1} - \tilde{\psi}_2(\partial\tilde{\chi}_3 - \tilde{\chi}_2\partial\tilde{\chi}_1)\,\mathrm{e}^{R_1+R_2}\\
J_{\lambda_1\cdot H} &= \tfrac{1}{3}(2\partial R_1 + \partial R_2) - \tilde{\psi}_1\partial\tilde{\chi}_1\,\mathrm{e}^{R_1} + (\tilde{\psi}_1\tilde{\psi}_2 - \tilde{\psi}_3)(\partial\tilde{\chi}_3 - \tilde{\chi}_2\partial\tilde{\chi}_1)\,\mathrm{e}^{R_1+R_2}\\
J_{\lambda_2\cdot H} &= \tfrac{1}{3}(\partial R_1 + 2\partial R_2) - \tilde{\psi}_2\partial\tilde{\chi}_2\,\mathrm{e}^{R_2} - \tilde{\psi}_3(\partial\tilde{\chi}_3 - \tilde{\chi}_2\partial\tilde{\chi}_1)\,\mathrm{e}^{R_1+R_2}\\
\bar{J}_{\alpha_1} &= \bar{\partial}\tilde{\chi}_1 - \tilde{\chi}_1^2\bar{\partial}\tilde{\psi}_1\,\mathrm{e}^{R_1} + \bar{\partial}\tilde{\psi}_2(\tilde{\chi}_1\tilde{\chi}_2 - \tilde{\chi}_3)\,\mathrm{e}^{R_2}\\
&\quad + (\bar{\partial}\tilde{\psi}_3 - \tilde{\psi}_2\bar{\partial}\tilde{\psi}_1)(\tilde{\chi}_1\tilde{\chi}_2 - \tilde{\chi}_3)\tilde{\chi}_1\,\mathrm{e}^{R_1+R_2} + \tilde{\chi}_1\partial R_1\\
\bar{J}_{-\alpha_1} &= \bar{\partial}\tilde{\psi}_1\,\mathrm{e}^{R_1} - \tilde{\chi}_2(\bar{\partial}\tilde{\psi}_3 - \tilde{\psi}_2\bar{\partial}\tilde{\psi}_1)\,\mathrm{e}^{R_1+R_2}\\
\bar{J}_{\lambda_1\cdot H} &= \tfrac{1}{3}(2\bar{\partial}R_1 + \bar{\partial}R_2) - \tilde{\chi}_1\bar{\partial}\tilde{\psi}_1\,\mathrm{e}^{R_1} + (\tilde{\chi}_1\tilde{\chi}_2 - \tilde{\chi}_3)(\bar{\partial}\tilde{\psi}_3 - \tilde{\psi}_2\bar{\partial}\tilde{\psi}_1)\,\mathrm{e}^{R_1+R_2}\\
\bar{J}_{\lambda_2\cdot H} &= \tfrac{1}{3}(\bar{\partial}R_1 + 2\bar{\partial}R_2) - \tilde{\chi}_2\bar{\partial}\tilde{\psi}_2\,\mathrm{e}^{R_2} - \tilde{\chi}_3(\bar{\partial}\tilde{\psi}_3 - \tilde{\psi}_2\bar{\partial}\tilde{\psi}_1)\,\mathrm{e}^{R_1+R_2}
\end{aligned}
\tag{3.31}
$$

where $\bar{\partial}J = \partial\bar{J} = 0$ and $J = J_{\lambda_1\cdot H}h_1 + J_{\lambda_2\cdot H}h_2 + \sum_{\alpha=\alpha_1,\alpha_2,\alpha_1+\alpha_2} J_\alpha E_{-\alpha} + J_{-\alpha}E_\alpha$.

One can easily verify that the algebra of the local (chiral) infinitesimal transformations (3.30), which leaves invariant the action of the ungauged IM (2.9) is $(SL(2)\otimes U(1))_{\text{left}}\otimes (SL(2)\otimes U(1))_{\text{right}}$.

3.2. *Global symmetries in the coset* G_0/G_0^0

The reduced model in the coset G_0/G_0^0 is obtained by implementing the additional constraints (2.12), i.e. by the vanishing of the chiral currents (2.12). For the $SL(3)$ example this allows the elimination of four degrees of freedom R_i, $\tilde{\psi}_1$ and $\tilde{\chi}_1$, i.e., taking into account (3.31) and (2.12) we find

$$\begin{aligned}
\partial R_1 &= 2\tilde{\psi}_1\partial\tilde{\chi}_1\,\mathrm{e}^{R_1} - \tilde{\psi}_2\partial\tilde{\chi}_2\,\mathrm{e}^{R_2} + (\partial\tilde{\chi}_3 - \tilde{\chi}_2\partial\tilde{\chi}_1)(\tilde{\psi}_3 - 2\tilde{\psi}_1\tilde{\psi}_2)\,\mathrm{e}^{R_1+R_2}\\
\partial R_2 &= -\tilde{\psi}_1\partial\tilde{\chi}_1\,\mathrm{e}^{R_1} + 2\tilde{\psi}_2\partial\tilde{\chi}_2\,\mathrm{e}^{R_2} + (\partial\tilde{\chi}_3 - \tilde{\chi}_2\partial\tilde{\chi}_1)(\tilde{\psi}_3 + \tilde{\psi}_1\tilde{\psi}_2)\,\mathrm{e}^{R_1+R_2}\\
\partial\tilde{\psi}_1 &= \tilde{\psi}_3\partial\tilde{\chi}_2\,\mathrm{e}^{R_2} \qquad \partial\tilde{\chi}_1 = \tilde{\psi}_2(\partial\tilde{\chi}_3 - \tilde{\chi}_2\partial\tilde{\chi}_1)\,\mathrm{e}^{R_2}\\
\bar{\partial} R_1 &= 2\tilde{\chi}_1\bar{\partial}\tilde{\psi}_1\,\mathrm{e}^{R_1} - \tilde{\chi}_2\bar{\partial}\tilde{\psi}_2\,\mathrm{e}^{R_2} + (\bar{\partial}\tilde{\psi}_3 - \tilde{\psi}_2\bar{\partial}\tilde{\psi}_1)(\tilde{\chi}_3 - 2\tilde{\chi}_1\tilde{\chi}_2)\,\mathrm{e}^{R_1+R_2}\\
\bar{\partial} R_2 &= -\tilde{\chi}_1\bar{\partial}\tilde{\psi}_1\,\mathrm{e}^{R_1} + 2\tilde{\chi}_2\bar{\partial}\tilde{\psi}_2\,\mathrm{e}^{R_2} + (\bar{\partial}\tilde{\psi}_3 - \tilde{\psi}_2\bar{\partial}\tilde{\psi}_1)(\tilde{\chi}_3 + \tilde{\chi}_1\tilde{\chi}_2)\,\mathrm{e}^{R_1+R_2}\\
\bar{\partial}\tilde{\psi}_1 &= \tilde{\chi}_2(\bar{\partial}\tilde{\psi}_3 - \tilde{\psi}_2\bar{\partial}\tilde{\psi}_1)\,\mathrm{e}^{R_2} \qquad \bar{\partial}\tilde{\chi}_1 = \tilde{\chi}_3\bar{\partial}\tilde{\psi}_2\,\mathrm{e}^{R_2}.
\end{aligned} \tag{3.32}$$

In order to eliminate the unphysical fields R_i, $\tilde{\psi}_1$ and $\tilde{\chi}_1$ we recall equation (2.19) relating the fields of the gauged and ungauged models (2.25) and (2.9), respectively. In terms of these variables the transformations (3.30) become

$$\begin{aligned}
\delta\psi_1 &= \tfrac{1}{2}(-\epsilon_1 - \epsilon_2 + \bar{\epsilon}_1 + \bar{\epsilon}_2)\psi_1 - \tfrac{1}{2}\epsilon_-\psi_1\tilde{\psi}_1 + \bar{\epsilon}_+\big(\psi_2\,\mathrm{e}^{-\frac{1}{2}R_1} + \tfrac{1}{2}\psi_1\tilde{\chi}_1\big)\\
\delta\chi_1 &= \tfrac{1}{2}(\epsilon_1 + \epsilon_2 - \bar{\epsilon}_1 - \bar{\epsilon}_2)\chi_1 - \tfrac{1}{2}\bar{\epsilon}_+\chi_1\tilde{\chi}_1 + \epsilon_-\big(\chi_2\,\mathrm{e}^{-\frac{1}{2}R_1} + \tfrac{1}{2}\chi_1\tilde{\psi}_1\big)\\
\delta\psi_2 &= \epsilon_-\big(\tfrac{1}{2}\psi_2\tilde{\psi}_1 - \psi_1\,\mathrm{e}^{-\frac{1}{2}R_1}\big) - \tfrac{1}{2}\bar{\epsilon}_+\psi_2\tilde{\chi}_1 + \tfrac{1}{2}(-\epsilon_2 + \bar{\epsilon}_2)\psi_2\\
\delta\chi_2 &= \bar{\epsilon}_+\big(\tfrac{1}{2}\chi_2\tilde{\chi}_1 - \chi_1\,\mathrm{e}^{-\frac{1}{2}R_1}\big) - \tfrac{1}{2}\epsilon_-\chi_2\tilde{\psi}_1 + \tfrac{1}{2}(\epsilon_2 - \bar{\epsilon}_2)\chi_2
\end{aligned} \tag{3.33}$$

where $\epsilon_i, \bar{\epsilon}_i, \epsilon_\pm$ and $\bar{\epsilon}_\pm$ now satisfy certain restrictions[2] (see equation (3.47) below) which forces them to be constants. By simplifying equation (3.32) we obtain the nonlocal fields R_i in the form

$$\begin{aligned}
\partial R_1 &= \frac{\psi_1\partial\chi_1}{\Delta}\left(1 + \frac{3}{2}\psi_2\chi_2\right) - \frac{\psi_2\partial\chi_2}{\Delta}\left(\Delta_2 + \frac{3}{2}\psi_1\chi_1\right)\\
\partial R_2 &= \frac{\psi_1\partial\chi_1}{\Delta} + \frac{\psi_2\partial\chi_2}{\Delta}\left(2\Delta_2 + \frac{3}{2}\psi_1\chi_1\right)\\
\bar{\partial} R_1 &= \frac{\chi_1\bar{\partial}\psi_1}{\Delta}\left(1 + \frac{3}{2}\psi_2\chi_2\right) - \frac{\chi_2\bar{\partial}\psi_2}{\Delta}\left(\Delta_2 + \frac{3}{2}\psi_1\chi_1\right)\\
\bar{\partial} R_2 &= \frac{\chi_1\bar{\partial}\psi_1}{\Delta} + \frac{\chi_2\bar{\partial}\psi_2}{\Delta}\left(2\Delta_2 + \frac{3}{2}\psi_1\chi_1\right)
\end{aligned} \tag{3.34}$$

[2] Coming from the requirement of the invariance of constraints equations (3.32).

where $\Delta = (1+\psi_2\chi_2)^2 + \psi_1\chi_1\left(1+\frac{3}{4}\psi_2\chi_2\right)$, $\Delta_2 = 1+\psi_2\chi_2$. In addition we find

$$\begin{aligned}
\partial\tilde{\chi}_1 &= \frac{\psi_2}{\Delta}\left(\partial\chi_1\Delta_2 - \frac{1}{2}\chi_1\psi_2\partial\chi_2\right)\mathrm{e}^{-\frac{1}{2}R_1} \\
\partial\tilde{\psi}_1 &= \frac{\psi_1}{\Delta}\left(\partial\chi_2(1+\psi_1\chi_1+\psi_2\chi_2) - \frac{1}{2}\chi_2\psi_1\partial\chi_1\right)\mathrm{e}^{-\frac{1}{2}R_1} \\
\bar{\partial}\tilde{\psi}_1 &= \frac{\chi_2}{\Delta}\left(\bar{\partial}\psi_1\Delta_2 - \frac{1}{2}\psi_1\chi_2\bar{\partial}\psi_2\right)\mathrm{e}^{-\frac{1}{2}R_1} \\
\bar{\partial}\tilde{\chi}_1 &= \frac{\chi_1}{\Delta}\left(\bar{\partial}\psi_2(1+\psi_1\chi_1+\psi_2\chi_2) - \frac{1}{2}\chi_1\psi_2\bar{\partial}\psi_1\right)\mathrm{e}^{-\frac{1}{2}R_1}.
\end{aligned} \tag{3.35}$$

We next define the conserved topological currents

$$j_{R_i,\mu} = \epsilon_{\mu\nu}\partial_\nu R_i \qquad i=1,2 \qquad j_{\tilde{\psi}_1,\mu} = \epsilon_{\mu\nu}\partial_\nu\tilde{\psi}_1 \qquad j_{\tilde{\chi}_1,\mu} = \epsilon_{\mu\nu}\partial_\nu\tilde{\chi}_1. \tag{3.36}$$

Using the equations of motion derived from (2.25), one can confirm the following conservation laws:

$$\bar{\partial}j = \partial\bar{j} \qquad j = j_{\tilde{\psi}_1} \qquad j_{\tilde{\chi}_1} \qquad j = j_{R_i} \qquad i=1,2 \tag{3.37}$$

where $j = \frac{1}{2}(j_0+j_1)$, $\bar{j} = \frac{1}{2}(j_0-j_1)$. Note that (3.34) and (3.35) define the nonlocal fields $R_1, R_2, \tilde{\psi}_1, \tilde{\chi}_1$ in terms of the physical fields ψ_1, ψ_2, χ_1 and χ_2. Hence, the conservation of the currents defined by the r.h.s. of (3.34) and (3.35) is non-trivial and requires the use of the equations of motion.

3.3. Algebra of the global symmetries

The simplest way to derive the algebra of symmetries of gauged IM (2.25) (generated by transformations (3.33)) is to realize the charges of *non-chiral* conserved currents (3.34), (3.35) and (3.36)

$$\begin{aligned}
Q_1 &= \frac{1}{3}\int(2\partial_x R_1 + \partial_x R_2)\,\mathrm{d}x \qquad & Q_2 &= \frac{1}{3}\int(\partial_x R_1 + 2\partial_x R_2)\,\mathrm{d}x \\
Q_{\tilde{\chi}_1} &= Q_- = \int\partial_x\tilde{\chi}_1\,\mathrm{d}x \qquad & Q_{\tilde{\psi}_1} &= Q_+ = \int\partial_x\tilde{\psi}_1\,\mathrm{d}x
\end{aligned} \tag{3.38}$$

in terms of the canonical momenta

$$\begin{aligned}
\Pi_{\psi_1} &= \frac{\delta\mathcal{L}}{\delta\dot{\psi}_1} = \frac{-k}{2\pi}\left(\frac{\partial\chi_1}{\Delta}(1+\psi_2\chi_2) - \frac{1}{2}\frac{\partial\chi_2}{\Delta}\chi_1\psi_2\right) \\
\Pi_{\psi_2} &= \frac{\delta\mathcal{L}}{\delta\dot{\psi}_2} = \frac{-k}{2\pi}\left(\frac{\partial\chi_2}{\Delta}(1+\psi_1\chi_1+\psi_2\chi_2) - \frac{1}{2}\frac{\partial\chi_1}{\Delta}\chi_2\psi_1\right) \\
\Pi_{\chi_1} &= \frac{\delta\mathcal{L}}{\delta\dot{\chi}_1} = \frac{-k}{2\pi}\left(\frac{\bar{\partial}\psi_1}{\Delta}(1+\psi_2\chi_2) - \frac{1}{2}\frac{\bar{\partial}\psi_2}{\Delta}\chi_2\psi_1\right) \\
\Pi_{\chi_2} &= \frac{\delta\mathcal{L}}{\delta\dot{\chi}_2} = \frac{-k}{2\pi}\left(\frac{\bar{\partial}\psi_2}{\Delta}(1+\psi_1\chi_1+\psi_2\chi_2) - \frac{1}{2}\frac{\bar{\partial}\psi_1}{\Delta}\chi_1\psi_2\right).
\end{aligned} \tag{3.39}$$

By substituting (3.39) in equations (3.34) and (3.35) we obtain

$$
\begin{aligned}
\partial R_1 &= \frac{-2\pi}{k}\left(\psi_1\Pi_{\psi_1} - \psi_2\Pi_{\psi_2}\right) & \bar{\partial} R_1 &= \frac{-2\pi}{k}\left(\chi_1\Pi_{\chi_1} - \chi_2\Pi_{\chi_2}\right)\\
\partial R_2 &= \frac{-2\pi}{k}\left(\psi_1\Pi_{\psi_1} + 2\psi_2\Pi_{\psi_2}\right) & \bar{\partial} R_2 &= \frac{-2\pi}{k}\left(\chi_1\Pi_{\chi_1} + 2\chi_2\Pi_{\chi_2}\right)\\
\partial \tilde{\chi}_1 &= \frac{-2\pi}{k}\psi_2\Pi_{\psi_1}\,\mathrm{e}^{-\frac{1}{2}R_1} & \bar{\partial} \tilde{\chi}_1 &= \frac{-2\pi}{k}\chi_1\Pi_{\chi_2}\,\mathrm{e}^{-\frac{1}{2}R_1}\\
\partial \tilde{\psi}_1 &= \frac{-2\pi}{k}\psi_1\Pi_{\psi_2}\,\mathrm{e}^{-\frac{1}{2}R_1} & \bar{\partial} \tilde{\psi}_1 &= \frac{-2\pi}{k}\chi_2\Pi_{\chi_1}\,\mathrm{e}^{-\frac{1}{2}R_1}.
\end{aligned} \tag{3.40}
$$

In order to calculate the field transformations

$$
\begin{aligned}
\delta_\pm\psi_i &= \{Q_\pm, \psi_i\}\epsilon^g_\pm & \delta_\pm\chi_i &= \{Q_\pm, \chi_i\}\epsilon^g_\pm\\
\delta_j\psi_i &= \{Q_j, \psi_i\}\epsilon^g_j & \delta_j\chi_i &= \{Q_j, \chi_i\}\epsilon^g_j
\end{aligned} \tag{3.41}
$$

we use the canonical Poisson brackets (PBs) (where ϵ^g are constant parameters)

$$
\left\{\Pi_{\phi_i}(x), \phi_k(y)\right\} = \delta_{ik}\delta(x-y) \qquad \phi_k = \psi_i, \chi_i \tag{3.42}
$$

and also a few consequences of (3.24) and (3.40)

$$
\begin{aligned}
\{\partial_x R_1(x), \psi_i(y)\} &= (-1)^{i+1}\psi_i(y)\delta(x-y) & \{\partial_x R_1(x), \chi_i(y)\} &= (-1)^{i}\chi_i(y)\delta(x-y)\\
\{\partial_x R_2(x), \psi_2(y)\} &= 2\psi_2(y)\delta(x-y) & \{\partial_x R_2(x), \chi_2(y)\} &= -2\chi_2(y)\delta(x-y)
\end{aligned} \tag{3.43}
$$

etc. Evaluating the corresponding PBs we find the field transformations we seek

$$
\begin{aligned}
\delta_+\chi_1 &= \tfrac{1}{2}\left(\chi_2\,\mathrm{e}^{-\frac{1}{2}R_1} + \tfrac{1}{2}\chi_1\tilde{\psi}_1\right)\epsilon^g_+ & \delta_-\chi_1 &= \tfrac{1}{4}\chi_1\tilde{\chi}_1\epsilon^g_-\\
\delta_+\psi_1 &= -\tfrac{1}{4}\psi_1\tilde{\psi}_1\epsilon^g_+ & \delta_-\psi_1 &= -\tfrac{1}{2}\left(\psi_2\,\mathrm{e}^{-\frac{1}{2}R_1} + \tfrac{1}{2}\psi_1\tilde{\chi}_1\right)\epsilon^g_-\\
\delta_+\chi_2 &= -\tfrac{1}{4}\chi_2\tilde{\psi}_1\epsilon^g_+ & \delta_-\chi_2 &= \tfrac{1}{2}\left(\chi_1\,\mathrm{e}^{-\frac{1}{2}R_1} - \tfrac{1}{2}\chi_2\tilde{\chi}_1\right)\epsilon^g_-\\
\delta_+\psi_2 &= -\tfrac{1}{2}\left(\psi_1\,\mathrm{e}^{-\frac{1}{2}R_1} - \tfrac{1}{2}\psi_2\tilde{\psi}_1\right)\epsilon^g_+ & \delta_-\psi_2 &= \tfrac{1}{4}\psi_2\tilde{\chi}_1\epsilon^g_-.
\end{aligned} \tag{3.44}
$$

Note that the above transformations are nonlocal due to the presence of $\tilde{\psi}_1$, $\tilde{\chi}_1$ and R_1 which are defined in terms of integrals of the fields ψ_i, χ_i and their derivatives

$$
\begin{aligned}
\tilde{\psi}_1(x) &= \frac{1}{2}\int \epsilon(x-y)(\psi_1(y)\Pi_{\psi_2}(y) - \chi_2(y)\Pi_{\chi_1}(y))\,\mathrm{e}^{-\frac{1}{2}R_1(y)}\,\mathrm{d}y\\
R_1(x) &= \frac{1}{2}\int \epsilon(x-y)(\psi_1(y)\Pi_{\psi_1}(y) - \psi_2(y)\Pi_{\psi_2}(y) - \chi_1(y)\Pi_{\chi_1}(y) + \chi_2(y)\Pi_{\chi_2}(y))\,\mathrm{d}y
\end{aligned} \tag{3.45}
$$

and $\tilde{\chi}_1 = \tilde{\psi}_1(\psi_1 \leftrightarrow \psi_2, \chi_1 \leftrightarrow \chi_2)$. Instead the transformations generated by the charges Q_1 and Q_2 have the following simple, local and linear in the fields form:

$$
\begin{aligned}
\delta_1\chi_1 &= -\chi_1\epsilon^g_1 & \delta_2\chi_1 &= -\chi_1\epsilon^g_2\\
\delta_1\psi_1 &= \psi_1\epsilon^g_1 & \delta_2\psi_1 &= \psi_1\epsilon^g_2\\
\delta_1\chi_2 &= 0 & \delta_2\chi_2 &= -\chi_2\epsilon^g_2\\
\delta_1\psi_2 &= 0 & \delta_2\psi_2 &= \psi_2\epsilon^g_2.
\end{aligned} \tag{3.46}
$$

Observe that the above transformations coincide precisely with the transformations (3.33) derived in section 3.2 provided the following identities take place:

$$
\begin{aligned}
2\epsilon^g_1 &= \bar{\epsilon}_1 - \epsilon_1 & 2\epsilon^g_2 &= \bar{\epsilon}_2 - \epsilon_2\\
\tfrac{1}{2}\epsilon^g_+ &= \epsilon_- & -\tfrac{1}{2}\epsilon^g_- &= \bar{\epsilon}_+.
\end{aligned} \tag{3.47}
$$

The PB algebra of the charges $Q_\pm, Q_1, Q_2$ can be evaluated with the help of equations (3.42), (3.43) and (3.45) yielding the following deformed structure:

$$\{Q_1, Q_\pm\} = \pm Q_\pm \qquad \{Q_2, Q_\pm\} = 0 \tag{3.48}$$
$$\{Q_+, Q_-\} = -\left(\frac{2\pi}{k}\right)^2 \int \partial_x \, \mathrm{e}^{-R_1}\, \mathrm{d}x = 2\kappa \left(\frac{2\pi}{k}\right)^2 \sinh\left(Q_1 - \frac{1}{2}Q_2\right)$$

where $\kappa = \exp\left(\frac{1}{2}(R_1(\infty) + R_1(-\infty))\right)$. Note that κ is a constant operator commuting with all the other generators. Finally, one can verify the invariance of the gauged IM (2.25) under the above nonlocal transformations by calculating the PBs of the charges $Q_\pm, Q_i$ with the Hamiltonian of the model

$$H = \int \mathrm{d}x((1 + \psi_2\chi_2)\Pi_{\chi_2}\Pi_{\psi_2} + \frac{1}{2}\psi_2\chi_1\Pi_{\chi_1}\Pi_{\psi_2} + (1 + \psi_1\chi_1 + \psi_2\chi_2)\Pi_{\chi_1}\Pi_{\psi_1}$$
$$+ \frac{1}{2}\psi_1\chi_2\Pi_{\chi_2}\Pi_{\psi_1} + \psi_1'\Pi_{\psi_1} + \psi_2'\Pi_{\psi_2} - \chi_1'\Pi_{\chi_1} - \chi_2'\Pi_{\chi_2} + V). \tag{3.49}$$

After a tedious but straightforward calculations we find that

$$\{Q_\pm, H\} = 0 \qquad \{Q_i, H\} = 0.$$

Hence the IM in consideration is invariant under the algebra (3.48), which after certain rescaling of the generators (see, for example, [26]) can be identified with the q-deformed $SL(2, R) \otimes U(1)$ PB algebra.

4. Dressing transformations and vertex operators

As is well known [6, 25] the dressing transformation and the vertex operators method represent a powerful tool for the construction of soliton solutions for the affine Toda models. Let us consider two arbitrary solutions $B_s \in \hat{G}_0$, $s = 1, 2$ of equations (2.7) written for the case of $A_2^{(1)}$ extended by d and the central term c, i.e.

$$B_s = g_{0s}\, \mathrm{e}^{\nu_s c + \eta_s d}.$$

The corresponding Lax (L-S) connections (2.27) $\mathcal{A}(s) = \mathcal{A}(B_s)$, $\bar{\mathcal{A}}(s) = \bar{\mathcal{A}}(B_s)$ are related by gauge (dressing) transformations $\theta_{-,+} = \exp \mathcal{G}_{<,>}$,

$$\mathcal{A}_\mu(2) = \theta_\pm \mathcal{A}_\mu(1)\theta_\pm^{-1} + \left(\partial_\mu \theta_\pm\right)\theta_\pm^{-1}. \tag{4.50}$$

They leave invariant the equations of motion (2.7) as well as the auxiliar linear problem, i.e. the pure gauge $\mathcal{A}_\mu$ defined in terms of the monodromy matrix $T(B_s)$,

$$(\partial_\mu - \mathcal{A}(B_s)_\mu)T_s(B_s) = 0. \tag{4.51}$$

The consistency of equations (4.50) and (4.51) imply the following relations:

$$T_2 = \theta_\pm T_1 \qquad \text{i.e.} \quad \theta_+ T_1 = \theta_- T_1 g^{(1)} \tag{4.52}$$

where $g^{(1)} \in \hat{G}$ is an arbitrary constant element of the corresponding affine group. Suppose $T_1 = T_0(B_{\text{vac}})$ is the vacuum solution,

$$\begin{aligned} &B_{\text{vac}}\epsilon_- B_{\text{vac}}^{-1} = \epsilon_- \qquad && \bar{\partial} B_{\text{vac}} B_{\text{vac}}^{-1} = \mu^2 z c \\ &\mathcal{A}_{\text{vac}} = -\epsilon_- && \bar{\mathcal{A}}_{\text{vac}} = \epsilon_+ + \mu^2 z c \end{aligned} \tag{4.53}$$

and $T_0 = \exp(-z\epsilon_-)\exp(\bar{z}\epsilon_+)$ as one can easily check by using the fact that $[\epsilon_+, \epsilon_-] = \mu^2 c$. According to equations (4.50) and (4.52), every solution $T_2 = T(B)$ can be obtained from vacuum configuration (4.53) by an appropriate gauge transformation $\theta_\pm$. In fact, equations (4.50) with $\mathcal{A}_{\text{vac}}$ and $\bar{\mathcal{A}}_{\text{vac}}$ as in equation (4.53) and

$$\mathcal{A}(B) = -B\epsilon_- B^{-1} \qquad \bar{\mathcal{A}}(B) = \epsilon_+ + \bar{\partial} B B^{-1}$$

allow us to derive $\theta_\pm$ as functionals of B, i.e. $\theta_\pm = \theta_\pm(B)$. We next apply equations (4.52),

$$\theta_-^{-1}\theta_+ = T_{\rm vac} g^{(1)} T_{\rm vac}^{-1} \tag{4.54}$$

in order to obtain a non-trivial field configuration B in terms of $g^{(1)} \in G$ and a certain highest weight (h.w.) representation of the algebra $A_2^{(1)}$ as we shall see below. The first step consists in substituting $\mathcal{A}_{\rm vac}$, $\bar{\mathcal{A}}_{\rm vac}$ and $\mathcal{A}(B)$, $\bar{\mathcal{A}}(B)$ in equation (4.50) and then solving it grade by grade remembering that $\theta_\pm$ may be decomposed in the form of infinite products

$$\theta_- = {\rm e}^{t(0)}\,{\rm e}^{t(-1)}\ldots \qquad \theta_+ = {\rm e}^{v(0)}\,{\rm e}^{v(1)}\ldots$$

where $t(-k)$ and $v(k)$, $k = 1, 2, \ldots$ denote linear combinations of grade $p = \mp k$ generators. For grade zero we find

$$t(0) = H(\bar{z}) \qquad {\rm e}^{v(0)} = B\,{\rm e}^{G(z)-\mu^2 z\bar{z}c}$$

where the arbitrary functions $H(\bar{z}), G(z) \in \mathcal{G}_0^0$ and are fixed to zero due to the subsidiary constraints (3.34), (3.35), i.e., $H(\bar{z}) = G(z) = 0$. The equations for $v(1)$, $t(-1)$ appear to be of the form

$$B^{-1}\partial B - \mu^2\bar{z}c = [v(1), \epsilon_-] \qquad \bar{\partial}BB^{-1} = [t(-1), \epsilon_+] + \mu^2 zc.$$

The next step is to consider certain matrix elements (taken for the h.w. representation $|\lambda_l\rangle$) of equation (4.54). Since $v(i)|\lambda_l\rangle = 0$ and $\langle\lambda_l|t(-i) = 0$, $i > 0$, we conclude that

$$\langle\lambda_l|B|\lambda_l\rangle\,{\rm e}^{-\mu^2 z\bar{z}} = \langle\lambda_l|T_0 g^{(1)} T_0^{-1}|\lambda_l\rangle. \tag{4.55}$$

Taking into account the explicit parametrization of the zero grade subgroup element $B = nam$ (2.8) in terms of the fields, ν, R_i, ψ_a, χ_a and choosing specific matrix elements we derive their explicit spacetime dependence,

$$\begin{aligned}
\tau_0 &\equiv {\rm e}^{\nu-\mu^2 z\bar{z}} = \langle\lambda_0|T_0 g^{(1)} T_0^{-1}|\lambda_0\rangle \\
\tau_1 &\equiv {\rm e}^{\frac{1}{3}(2R_1+R_2)+\nu-\mu^2 z\bar{z}} = \langle\lambda_1|T_0 g^{(1)} T_0^{-1}|\lambda_1\rangle \\
\tau_2 &\equiv {\rm e}^{\frac{1}{3}(R_1+2R_2)+\nu-\mu^2 z\bar{z}} = \langle\lambda_2|T_0 g^{(1)} T_0^{-1}|\lambda_2\rangle \\
\tau_{\psi_3} &\equiv {\rm e}^{\frac{1}{3}(2R_1+R_2)+\nu-\mu^2 z\bar{z}}\tilde{\psi}_3 = \langle\lambda_1|T_0 g^{(1)} T_0^{-1} E^{(0)}_{-\alpha_1-\alpha_2}|\lambda_1\rangle \\
\tau_{\chi_3} &\equiv {\rm e}^{\frac{1}{3}(2R_1+R_2)+\nu-\mu^2 z\bar{z}}\tilde{\chi}_3 = \langle\lambda_1|E^{(0)}_{\alpha_1+\alpha_2} T_0 g^{(1)} T_0^{-1}|\lambda_1\rangle \\
\tau_{\psi_2} &\equiv {\rm e}^{\frac{1}{3}(R_1+2R_2)+\nu-\mu^2 z\bar{z}}\tilde{\psi}_2 = \langle\lambda_2|T_0 g^{(1)} T_0^{-1} E^{(0)}_{-\alpha_2}|\lambda_2\rangle \\
\tau_{\chi_2} &\equiv {\rm e}^{\frac{1}{3}(R_1+2R_2)+\nu-\mu^2 z\bar{z}}\tilde{\chi}_2 = \langle\lambda_2|E^{(0)}_{\alpha_2} T_0 g^{(1)} T_0^{-1}|\lambda_2\rangle \\
\tau_{\psi_1} &\equiv {\rm e}^{\frac{1}{3}(2R_1+R_2)+\nu-\mu^2 z\bar{z}}\tilde{\psi}_1 = \langle\lambda_1|T_0 g^{(1)} T_0^{-1} E^{(0)}_{-\alpha_1}|\lambda_1\rangle \\
\tau_{\chi_1} &\equiv {\rm e}^{\frac{1}{3}(2R_1+R_2)+\nu-\mu^2 z\bar{z}}\tilde{\chi}_1 = \langle\lambda_1|E^{(0)}_{\alpha_1} T_0 g^{(1)} T_0^{-1}|\lambda_1\rangle.
\end{aligned} \tag{4.56}$$

In order to make the construction of solution (4.56) complete it remains to specify the constant affine group element $g^{(1)}$, which encodes the information (including topological properties) about the N-soliton structure of equations (2.7).

Since $\epsilon_\pm$ form a Heisenberg subalgebra, $[\epsilon_+, \epsilon_-] = \mu^2 c$ and we have to calculate the matrix elements (τ-functions), say

$$\langle\lambda_0|\,{\rm e}^{-z\epsilon_-}\,{\rm e}^{\bar{z}\epsilon_+} g^{(1)}\,{\rm e}^{-\bar{z}\epsilon_+}\,{\rm e}^{z\epsilon_-}|\lambda_0\rangle$$

it is instructive to introduce the eigenvectors ($F(\gamma)$) of $\epsilon_\pm$, i.e.,

$$[\epsilon^\pm, F(\gamma)] = f^\pm(\gamma)F(\gamma). \tag{4.57}$$

Following [24] (see also [11]) we find four non-trivial types of eigenvectors

$$F_{\pm}(\gamma) = \sum_{n\in Z} \left(E^{(n)}_{\pm\alpha_2} + E^{(n)}_{\pm(\alpha_1+\alpha_2)}\right)\gamma^{-n} \qquad \tilde{F}_{\pm}(\gamma) = \sum_{n\in Z} \left(E^{(n)}_{\pm\alpha_2} - E^{(n)}_{\pm(\alpha_1+\alpha_2)}\right)\gamma^{-n} \tag{4.58}$$

as well as the trivial eigenvectors

$$F^0_{\pm}(\gamma) = \sum_{n\in Z} E^{(n)}_{\pm\alpha_1}\gamma^{-n}. \tag{4.59}$$

Their eigenvalues are given by

$$\begin{aligned}
&[\epsilon_+, F_{\pm}(\gamma)] = \pm\mu\gamma F_{\pm}(\gamma) \qquad [\epsilon_+, \tilde{F}_{\pm}(\gamma)] = \pm\mu\gamma \tilde{F}_{\pm}(\gamma) \qquad [\epsilon_+, F^0_{\pm}(\gamma)] = 0\\
&[\epsilon_-, F_{\pm}(\gamma)] = \pm\mu\gamma^{-1} F_{\pm}(\gamma) \qquad [\epsilon_-, \tilde{F}_{\pm}(\gamma)] = \pm\mu\gamma^{-1} \tilde{F}_{\pm}(\gamma) \qquad [\epsilon_-, F^0_{\pm}(\gamma)] = 0.
\end{aligned} \tag{4.60}$$

Note that $F_{\pm}(\gamma)$, $\tilde{F}_{\pm}(\gamma)$ and $F^0_{\pm}(\gamma)$ together with

$$\lambda_i \cdot H(\gamma) = \sum_{n\in Z} \lambda_i \cdot H^{(n)}\gamma^{-n} \qquad i = 1, 2 \tag{4.61}$$

form a new basis for the affine $A^{(1)}_2$ algebra. In this basis, we define the affine group element $g^{(1)}$ as

$$g^{(1)} = \prod_a \mathrm{e}^{d_a F_a(\gamma)} \qquad F_a(\gamma) = \{F_{\pm}, \tilde{F}_{\pm}, \lambda_i \cdot H(\gamma), i = 1, 2, F^0_{\pm}(\gamma)\}$$

with the property

$$\begin{aligned}
&T_0 g^{(1)} T_0^{-1} = \exp\left(\sum_a d_a \rho_a(\gamma) F_a(\gamma)\right) = \prod_a (1 + d_a \rho_a(\gamma) F_a(\gamma))\\
&\rho_a(\gamma) = \exp\left(-z f^-_a(\gamma) + \bar{z} f^+_a(\gamma)\right).
\end{aligned} \tag{4.62}$$

The use of this basis drastically simplifies the calculation of the τ-functions (4.56). Note that in the above formula each $F_a^2 = 0$, but the mixed terms $F_a F_b$ do contribute [24].

5. Two-vertex soliton solutions

An important question concerns the specific choice of the form of $g^{(1)}$ that leads to different species of neutral and charged solitons and breathers. As in the case of the complex sine-Gordon model the 1-soliton solutions can be constructed in terms of two-vertex operators, i.e., $g^{(1)}$ chosen in one of the following forms:

$$g^{(1)}(\gamma_1, \gamma_2) = \mathrm{e}^{d_1 F_+(\gamma_1)}\, \mathrm{e}^{d_2 F_-(\gamma_2)} \tag{5.63}$$

$$\tilde{g}^{(1)}(\gamma_1, \gamma_2) = \mathrm{e}^{\tilde{d}_1 \tilde{F}_+(\gamma_1)}\, \mathrm{e}^{\tilde{d}_2 \tilde{F}_-(\gamma_2)} \tag{5.64}$$

$$g^{(1)}_{01}(\gamma_1, \gamma_2) = \mathrm{e}^{d_{01} F_+(\gamma_1)}\, \mathrm{e}^{\tilde{d}_{01} \tilde{F}_-(\gamma_2)} \tag{5.65}$$

$$g^{(1)}_{02}(\gamma_1, \gamma_2) = \mathrm{e}^{\tilde{d}_{02} \tilde{F}_+(\gamma_1)}\, \mathrm{e}^{d_{02} F_-(\gamma_2)}. \tag{5.66}$$

For the case given by equation (5.63) according to (4.62) we find for the τ -functions (4.56)

$$\begin{aligned}
\tau_0 &= 1 + 2\, d_1\, d_2 \rho_1(\gamma_1)\rho_2(\gamma_2) \frac{\gamma_1\gamma_2}{(\gamma_1 - \gamma_2)^2}\\
\tau_1 &= 1 + d_1\, d_2 \rho_1(\gamma_1)\rho_2(\gamma_2) \frac{\gamma_1(\gamma_1 + \gamma_2)}{(\gamma_1 - \gamma_2)^2}
\end{aligned}$$

$$\tau_2 = 1 + 2\, d_1\, d_2 \rho_1(\gamma_1)\rho_2(\gamma_2)\frac{\gamma_1^2}{(\gamma_1-\gamma_2)^2}$$
$$\tau_{\psi_3} = \tau_{\psi_2} = d_1\rho_1(\gamma_1) \qquad \tau_{\chi_3} = \tau_{\chi_2} = d_2\rho_2(\gamma_2)$$
$$\tau_{\psi_1} = \tau_{\chi_1} = d_1\, d_2\rho_1(\gamma_1)\rho_2(\gamma_2)\frac{\gamma_1}{\gamma_1-\gamma_2} \tag{5.67}$$

where $\rho_1(\gamma_1) = \exp\left(-\frac{z}{\gamma_1} + \bar{z}\gamma_1\right)$ and $\rho_2(\gamma_2) = \exp\left(\frac{z}{\gamma_2} - \bar{z}\gamma_2\right)$. In order to ensure the reality (and positivity) of the energy of the solution, we require the product $\rho_1(\gamma_1)\rho_2(\gamma_2)$ to be real. This leads to the following parametrization for γ_i:

$$\gamma_1 = -\mathrm{e}^{b-\mathrm{i}\alpha} \qquad \gamma_2 = \mathrm{e}^{b+\mathrm{i}\alpha} \qquad b, \alpha \in R$$

and for $\rho_i(\gamma_i)$ we obtain

$$\rho_1 = \mathrm{e}^{F+\mathrm{i}G} \qquad \rho_2 = \mathrm{e}^{F-\mathrm{i}G} \qquad z = \tfrac{1}{2}(x+t) \qquad \bar{z} = \tfrac{1}{2}(t-x)$$
$$F = \mu\cos(\alpha)[-t\sinh(b) + x\cosh(b)] \qquad G = \mu\sin(\alpha)[t\cosh(b) - x\sinh(b)].$$

It is convenient to choose the arbitrary complex constants d_1 and d_2 in the form

$$d_1 = \frac{\gamma_1-\gamma_2}{\sqrt{2\gamma_1\gamma_2}}\,\mathrm{e}^{\mathrm{i}\theta-\mu Y\cos(\alpha)\cosh(b)} \qquad d_2 = \frac{\gamma_1-\gamma_2}{\sqrt{2\gamma_1\gamma_2}}\,\mathrm{e}^{-\mathrm{i}\theta-\mu Y\cos(\alpha)\cosh(b)}$$

where θ and Y are new arbitrary real constants. Then the 1-soliton solutions corresponding to two-vertex $g^{(1)}$ (5.63) take the following simple form:

$$\begin{aligned}
&\mathrm{e}^{\nu-\mu^2 z\bar{z}} = 1 + \mathrm{e}^{2\tilde{F}} \\
&\mathrm{e}^{\frac{1}{3}(2R_1+R_2)} = \frac{\mathrm{e}^{-\tilde{F}} + \frac{1}{2}(1-\Gamma_1)\,\mathrm{e}^{\tilde{F}}}{\mathrm{e}^{\tilde{F}} + \mathrm{e}^{-\tilde{F}}} \qquad \Gamma_1 = \mathrm{e}^{-2\mathrm{i}\alpha} \\
&\mathrm{e}^{\frac{1}{3}(R_1+2R_2)} = \frac{\mathrm{e}^{-\tilde{F}} - \Gamma_1\,\mathrm{e}^{\tilde{F}}}{\mathrm{e}^{\tilde{F}} + \mathrm{e}^{-\tilde{F}}} \qquad \Gamma_2 = \mathrm{d}_1\,\mathrm{e}^{-\mathrm{i}\theta+\mu Y\cos(\alpha)\cosh(b)} = \frac{\gamma_1-\gamma_2}{\sqrt{2\gamma_1\gamma_2}} \\
&\psi_1 = \frac{\Gamma_2\,\mathrm{e}^{\mathrm{i}(G+\theta)}}{(\mathrm{e}^{\tilde{F}} + \mathrm{e}^{-\tilde{F}})}\left(\frac{\mathrm{e}^{-\tilde{F}} - \Gamma_1\,\mathrm{e}^{\tilde{F}}}{\mathrm{e}^{-\tilde{F}} + \frac{1}{2}(1-\Gamma_1)\,\mathrm{e}^{\tilde{F}}}\right)^{\frac{1}{2}} \\
&\chi_1 = \frac{\Gamma_2\,\mathrm{e}^{-\mathrm{i}(G+\theta)}}{(\mathrm{e}^{\tilde{F}} + \mathrm{e}^{-\tilde{F}})}\left(\frac{\mathrm{e}^{-\tilde{F}} - \Gamma_1\,\mathrm{e}^{\tilde{F}}}{\mathrm{e}^{-\tilde{F}} + \frac{1}{2}(1-\Gamma_1)\,\mathrm{e}^{\tilde{F}}}\right)^{\frac{1}{2}} \\
&\psi_2 = \frac{\Gamma_2\,\mathrm{e}^{\mathrm{i}(G+\theta)}}{(\mathrm{e}^{\tilde{F}} + \mathrm{e}^{-\tilde{F}})}\left(\frac{\mathrm{e}^{-\tilde{F}} + \mathrm{e}^{\tilde{F}}}{\mathrm{e}^{-\tilde{F}} + \frac{1}{2}(1-\Gamma_1)\,\mathrm{e}^{\tilde{F}}}\right)^{\frac{1}{2}} \\
&\chi_2 = \frac{\Gamma_2\,\mathrm{e}^{-\mathrm{i}(G+\theta)}}{(\mathrm{e}^{\tilde{F}} + \mathrm{e}^{-\tilde{F}})}\left(\frac{\mathrm{e}^{-\tilde{F}} + \mathrm{e}^{\tilde{F}}}{\mathrm{e}^{-\tilde{F}} + \frac{1}{2}(1-\Gamma_1)\,\mathrm{e}^{\tilde{F}}}\right)^{\frac{1}{2}}
\end{aligned} \tag{5.68}$$

where $\tilde{F}(t,x) = F(t, x-Y)$. The nonlocal fields $\tilde{\psi}_1$ and $\tilde{\chi}_1$ (whose asymptotics are important for determining the charges $Q_\pm$) are given by

$$\tilde{\psi}_1 = \tilde{\chi}_1 = -\frac{(1+\Gamma_1)\,\mathrm{e}^{2\tilde{F}}}{2\left(1+\frac{1}{2}(1-\Gamma_1)\,\mathrm{e}^{2\tilde{F}}\right)}. \tag{5.69}$$

As well known [6, 7, 10] the energy of such a solution is related to the asymptotics of $\ln\tau_0$, i.e.,

$$M = E(b=0) = \int_{-\infty}^{\infty}\mathrm{d}x\,T_{00} = -\frac{2}{\beta^2}\int_{-\infty}^{\infty}\mathrm{d}x\,\partial_x\ln\tau_0 = \frac{4\mu}{\beta^2}\cos(\alpha) \qquad \beta^2 = \frac{2\pi}{k}$$

in the rest frame $b = 0$. The corresponding Noether charges Q_1 and Q_2 (see equations (3.38)) are defined by the asymptotics of the nonlocal fields R_i

$$Q_1 = -\mathrm{i}\,(\alpha + \pi + \mathrm{i}\,\ln(\cos(\alpha))) \qquad Q_2 = -\mathrm{i}(\alpha + \pi + \mathrm{i}\ln(2\cos(\alpha))). \tag{5.70}$$

Similarly, for the charges $Q_\pm$ we find

$$Q_+ = Q_- = \tilde{\psi}_1(\infty) - \tilde{\psi}_1(-\infty) = \frac{\Gamma_1 + 1}{\Gamma_1 - 1} = \mathrm{i}\cot g(\alpha).$$

Therefore, the spectrum of the above 1-soliton solutions, i.e., M, Q_1, Q_2 and $Q_\pm$, is determined by the real constant α only.

The case when $g^{(1)}$ is taken in the form (5.64) is quite similar to that considered above. The only difference is that now we have

$$\tau_{\psi_3} = -\tau_{\psi_2} \qquad \tau_{\chi_3} = -\tau_{\chi_2} \qquad \tau_{\psi_1} = \tau_{\chi_1} = -\tilde{d}_1\tilde{d}_2\rho_1\rho_2\frac{\gamma_1}{\gamma_1 - \gamma_2}$$

and all the other τ-functions remain unchanged. As a consequence, the mass $\tilde{M}$ and charges $\tilde{Q}_1$, $\tilde{Q}_2$ are the same and $\tilde{Q}_\pm = -Q_\pm$. The cases (5.65) and (5.66) are quite different. We find that

$$\tau_0 = \tau_2 = 1 \qquad \tau_1 = 1 - d_1\tilde{d}_2\rho_1(\gamma_1)\rho_2(\gamma_2)\frac{\gamma_1}{\gamma_1 - \gamma_2}$$
$$\tau_{\psi_1} = -\tau_{\chi_1} = d_1\tilde{d}_2\rho_1(\gamma_1)\rho_2(\gamma_2)\frac{\gamma_1}{\gamma_1 - \gamma_2}$$
$$\tau_{\psi_3} = -\tau_{\psi_2} = d_1\rho_1(\gamma_1) \qquad \tau_{\chi_3} = -\tau_{\chi_2} = -\tilde{d}_2\rho_2(\gamma_2).$$

Such a solution has vanishing mass and charge $Q_2 = 0$.

6. Concluding remarks

Our analysis of the symmetries and 1-soliton solutions of the IM (2.25) leaves a few interesting open problems:

- How to construct more general 1-soliton solutions whose spectrum, M, Q_1, Q_2 and $Q_\pm$, is parametrized by four real parameters instead of one α as in equation (5.68).
- What are the symmetry properties of the 1-solitons (5.68), i.e. how to recognize the representations of the q-deformed algebra (3.48) to which these solitons belong.
- About the topological stability of these solitons and of the related strong-coupling particles of the IM (2.25).

The complete structure of the solitons (and particles) spectrum of such IM indeed requires answers to the above questions.

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